Escape problem under stochastic volatility: The Heston model

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We solve the escape problem for the Heston random diffusion model from a finite interval of span L. We obtain exact expressions for the survival probability (which amounts to solving the complete escape problem) as well as for the mean exit time. We also average the volatility in order to work out the problem for the return alone regardless of volatility. We consider these results in terms of the dimensionless normal level of volatility—a ratio of the three parameters that appear in the Heston model—and analyze their form in several asymptotic limits. Thus, for instance, we show that the mean exit time grows quadratically with large spans while for small spans the growth is systematically slower, depending on the value of the normal level. We compare our results with those of the Wiener process and show that the assumption of stochastic volatility, in an apparently paradoxical way, increases survival and prolongs the escape time. We finally observe that the model is able to describe the main exit-time statistics of the Dow-Jones daily index.

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I. INTRODUCTION

Models of financial dynamics based on two-dimensional diffusion processes, known as stochastic volatility (SV) models [1], are being widely accepted as a reasonable explanation for many empirical observations collected under the name of "stylized facts" [2]. In such models the volatility, that is, the standard deviation of returns originally thought to be a constant, is a random process coupled with the return so that they both form a two-dimensional diffusion process governed by a pair of Langevin equations [1].

Volatility is nowadays a key concept in any financial setting. It is the backbone of many financial products that are designed to cover investors' risk. Extreme values associated with volatility have thus a special meaning, as they do in physics and natural sciences where escape problems in noisy environments such as the Kramers problem are of the utmost importance [3,4].

Extreme-value problems have a clear financial interest apart from the obvious relation to the classic ruin problem. As an example, among others, let us mention the so-called leverage certificates (LC's) which are structured products offering a nonzero payoff only if the underlying asset does not escape from a pre-established domain over a certain time window [5]. Although usually sold as products insensitive to volatility changes, LC's are very sensitive to other risk components such as skewness and kurtosis [5]. On the other hand, stochastic volatility models result in fat tailed distributions for the return and show clustering in the volatility, two well-established facts in empirical data which are closely related to skewness and kurtosis [2]. For this reason, the solution to the escape problem under stochastic volatility can be used to derive a more precise price than that of the Wiener process for a wide class of LC products (see the recent works [6,7] for some alternative approaches to option pricing under stochastic volatility).

In a recent paper we have addressed a partial aspect of the problem: that of extreme times for the volatility regardless the value of the return [8]. Now we want to address the overall escape problem associated with both return and volatility. This is certainly a more difficult task because the return strongly depends on volatility while, in the standard approach to SV models, the latter is supposed to be independent of the former.

We are thus left with a two-dimensional escape problem which is always quite involved. The situation is similar to that of the unbounded Brownian particle where the extreme-value problem for the velocity of the particle is relatively easy to handle, while that of its position is much more intricate [3,9–11].

The extreme-time problem of the return has been addressed, to our knowledge, in only a few works. We refer the reader to our recent work on the subject [12,13], although it is based on the continuous-time random walk technique, which is an entirely different frame, and with a different scope, from that of SV models. Within the setting of the latter, we are only aware of the recent works by Bonanno *et al.* [14–17] where an approach to the hitting-time problem is addressed through numerical simulations of the Heston model and some variations of it (see also the work of Jafari *et al.* [18] for an empirical study of level crossing and hitting times).

In this paper we study the complete escape problem of one particular SV model: the Heston model [19]. Different SV models basically differ in the way the volatility depends on the underlying noise governing its dynamics. The Heston model has the benefit, over other SV models of allowing exact analytical developments. This is the case of its unrestricted (i.e., barrier-free) probability density function, which was obtained by Yakovenko and Dragulescu a few years ago [20] (see also Ref. [21]). Herein, we will obtain not only the exact expression of the mean escape time (MET) but the exact survival probability as well. The knowledge of the latter is equivalent to solving the entire escape problem.

The paper is organized as follows. In Sec. II we present the Heston model and obtain the complete solution to the escape problem. In Sec. III we evaluate the mean escape time

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and analyze its behavior for high and low volatility. In Sec. IV we average out the volatility assuming it has reached the stationary state. This allows us to get exact expressions for the survival probability and the mean escape time of the return alone. Conclusions are drawn in Sec. V, and some more technical details are in the Appendixes.

II. THE HESTON MODEL AND THE SURVIVAL PROBABILITY

Let $P(t)$ be a speculative price or the value of a financial index. We define the zero-mean return $X(t)$ through the stochastic differential (in the Itô sense)

$$dX(t) = \frac{dP(t)}{P(t)} - \left\langle \frac{dP(t)}{P(t)} \right\rangle, \tag{1}$$

where $\langle\cdot\rangle$ denotes the average [22]. In terms of $X(t)$ the Heston model [19] is a two-dimensional diffusion process $(X(t),Y(t))$ described by the following pair of stochastic differential equations (again, in the Itô sense):

$$dX(t) = \sqrt{Y(t)}dW_1(t), \tag{2}$$

$$dY(t) = -\alpha[Y(t) - m^2]dt + k\sqrt{Y(t)}dW_2(t), \tag{3}$$

where $W_i(t)$ are Wiener processes, i.e., $dW_i(t)=\xi_i(t)dt$ $(i=1,2)$, where $\xi_i(t)$ are zero-mean Gaussian white noises with $\langle \xi_i(t)\xi_i(t')\rangle = \delta_{ij}\delta(t-t')$ [23]. Note that in this particular model the volatility is

$$\sigma(t) = \sqrt{Y(t)}, \tag{4}$$

i.e., $Y(t)$ is the variance of return although, as long as no confusion arises, we will use the term "volatility variable" or just "volatility" for the random process $Y(t)$. In Eq. (3) the parameter m is the so-called normal level of volatility, $\alpha > 0$ is related to the "reverting force" toward the normal level $m \neq 0$ (see below) and k, sometimes referred to as the "vol of vol," measures the fluctuations of the volatility.

In the context of biological diffusion problems the process $Y(t)$ described by Eq. (3) was proposed many years ago by Feller [24] who, among other properties, proved that $Y(t)$ is always positive so that the volatility, Eq. (4), is real, positive, and well defined. This feature, along with a non-negligible (and exponential) autocorrelation with characteristic time $1/\alpha$, makes the process very appealing from the perspective of mathematical finance.

In 1985 Cox, Ingersoll, and Ross [25] introduced the same dynamics in connection with interest rates of bonds. Almost a decade later and aiming to provide a more realistic price for options, Heston [19] undertook the same dynamics but for the diffusion coefficient of financial price fluctuations as is precisely shown in Eqs. (2) and (3).

The resulting process has become quite popular among financial practitioners who want to include the effect of volatility changes in option pricing. Part of this success is due to the easy interpretation of the parameters. As mentioned, $1/\alpha$ provides the typical time that the volatility needs to reach the stationary state (the stationary density is the Gamma distribution; see Sec. IV). For this reason, α can also be interpreted as the strength of the reverting force that ties the process $Y(t)$ to its normal level m^2, the latter being the mean value of $Y(t)$ in the stationary state. Since for the Gamma distribution the stationary mean value cannot be zero, we conclude that $m \neq 0$. Finally, the magnitude of the volatility fluctuations is provided by k, which like α and m^2, has units of 1/(time).

Our main interest is the escape problem associated with the Heston model. To this end, let us denote by $S(x,y,t)$ the probability that the zero-mean return $X(t)$, starting at $X(0)=x$ with volatility $Y(0)=y$, is at time t inside the interval $(-L/2, L/2)$ without having ever left it during previous times. In other words, $S(x,y,t)$ is the survival probability (SP) for the joint process $(X(t),Y(t))$ to be at time t inside the strip

$$-L/2 \le X(t) \le L/2, \quad 0 < Y(t) < \infty,$$

with $X(0)=x$ and $Y(0)=y$.

The SP obeys the following backward Fokker-Planck equation [26]:

$$\frac{\partial S}{\partial t} = -\alpha(y-m^2)\frac{\partial S}{\partial y} + \frac{1}{2}k^2 y \frac{\partial^2 S}{\partial y^2} + \frac{1}{2} y \frac{\partial^2 S}{\partial x^2}, \tag{5}$$

with initial and boundary conditions, respectively, given by

$$S(x,y,0) = 1, \quad S(\pm L/2, y, t) = 0. \tag{6}$$

This problem can be solved by means of Fourier series. Indeed the boundary conditions, $S(\pm L/2, y, t)=0$, lead us to look for a solution of the form

$$S(x,y,t) = \sum_{n=0}^{\infty} S_n(y,t)\cos[(2n+1)\pi x/L], \tag{7}$$

where the Fourier coefficients $S_n(y,t)$ are

$$S_n(y,t) = \frac{2}{L}\int_{-L/2}^{L/2} S(x,y,t)\cos[(2n+1)\pi x/L]dx. \tag{8}$$

From Eqs. (5) and (8) we see that these coefficients are the solution to the initial-value problem

$$\frac{\partial S_n}{\partial t} = -\alpha(y-m^2)\frac{\partial S_n}{\partial y} + \frac{1}{2}k^2 y \frac{\partial^2 S_n}{\partial y^2} - \frac{1}{2}[(2n+1)\pi/L]^2 y S_n, \tag{9}$$

with initial condition

$$S_n(y,0) = \gamma_n, \tag{10}$$

where

$$\gamma_n = \frac{4(-1)^n}{\pi(2n+1)}. \tag{11}$$

Defining a new time scale τ and a new volatility variable v by the change of scale

$$\tau = \alpha t, \quad v = (2\alpha/k^2)y, \tag{12}$$

the problem above reads

$$\frac{\partial S_n}{\partial \tau} = -(v-\theta)\frac{\partial S_n}{\partial v} + v\frac{\partial^2 S_n}{\partial v^2} - (\beta_n/2L)^2 v S_n \tag{13}$$

and

$$S_n(v,0) = \gamma_n, \tag{14}$$

where

$$\beta_n = (k/\alpha)(2n+1)\pi \tag{15}$$

and

$$\theta = (2\alpha/k^2)m^2. \tag{16}$$

Before proceeding further let us remark that the parameter θ, which turns out to be crucial for the escape problem at hand, can be regarded as the "dimensionless normal level" of volatility. It represents a balance between the tendency toward the normal level measured by αm^2 and the volatility fluctuations quantified by k^2 (see the discussion in Sec. IV B regarding the cases $\theta<1$ and $\theta>1$).

The problem posed by Eqs. (13) and (14) is solved by the function

$$S_n(v,\tau) = \gamma_n \exp[-A_n(\tau) - B_n(\tau)v], \tag{17}$$

where $A_n(\tau)$ and $B_n(\tau)$ are functions of time to be determined. In effect, plugging it into Eq. (13), we see that Eq. (17) is the solution to the problem provided that

$$A_n(\tau) = \theta\int_0^\tau B_n(s)ds, \tag{18}$$

and $B_n(\tau)$ obeys the Riccati equation

$$\dot{B}_n = -B_n - B_n^2 + (\beta_n/2L)^2, \tag{19}$$

with initial condition $B_n(0)=0$.

In Appendix A we show that

$$A_n(\tau) = \theta\left[\mu_-\tau + \ln\left(\frac{\mu_+ + \mu_- e^{-\Delta_n\tau}}{\Delta_n}\right)\right] \tag{20}$$

and

$$B_n(\tau) = \mu_-\frac{1-e^{-\Delta_n\tau}}{1+(\mu_-/\mu_+)e^{-\Delta_n\tau}}, \tag{21}$$

where

$$\Delta_n = \sqrt{1+(\beta_n/L)^2}, \quad \mu_\pm = (\Delta_n \pm 1)/2. \tag{22}$$

Therefore, the solution to the escape problem for the two-dimensional Heston SV model is

$$S(x,v,\tau) = \sum_{n=0}^{\infty} \gamma_n \exp[-A_n(\tau) - B_n(\tau)v]\cos[(2n+1)\pi x/L]. \tag{23}$$

Figure 1 shows in a three-dimensional plot this SP as a function of the return x and volatility variable v for $\tau=0.1$ [27].

In the asymptotic regime, for either long or short times, the SP is somewhat simpler. Thus, when $\tau \gg 1$ (i.e., $t\gg\alpha^{-1}$), we have $A_n(\tau)\simeq\theta\mu_-\tau$ and $B_n(\tau)\simeq\mu_-$. Hence,

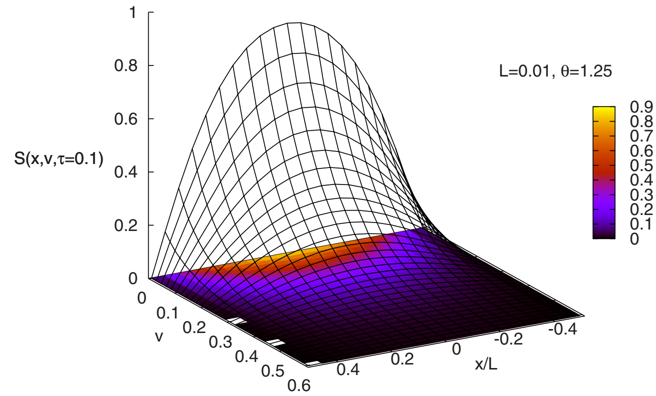

FIG. 1. (Color online) Survival probability $S(x,v,\tau)$ given by Eq. (23) with $\tau=0.1$ ($t=2.22$ days) and $L=0.01$ in terms of return x and volatility v. Parameters of the model: $\theta=1.25$, $\alpha=0.045$ day^{-1}, $m=0.093$ day$^{-1/2}$, and $k=0.0014$ day^{-1}. Recall Eq. (16) and note that there exist only three independent parameters.

$$S(x,v,\tau) \simeq \sum_{n=0}^{\infty} e^{-\mu_-(\theta\tau+v)}\cos[(2n+1)\pi x/L] \quad (\tau\gg 1). \tag{24}$$

On the other hand, for short times $\tau\ll 1$ (i.e., $t\ll\alpha^{-1}$) we write $e^{-\Delta_n\tau}=1-\Delta_n\tau+O(\tau^2)$ and taking into account that $\mu_-+\mu_+=\Delta_n$ and $\mu_-\mu_+=-(\beta_n/2L)^2$, we see from Eqs. (20) and (21) that

$$A_n = \theta[\mu_-\tau + \ln(1-\mu_-\tau)] + O(\tau^2),$$

$$B_n = \frac{(\beta_n/2L)^2\tau}{1-\mu_-\tau} + O(\tau^2),$$

whence,

$$S(x,v,\tau) \simeq \sum_{n=0}^{\infty}\frac{1}{(1-\mu_-\tau)^\theta}\exp\left[-\left(\theta\mu_- + \frac{(\beta_n/2L)^2 v}{1-\mu_-\tau}\right)\tau\right] \times\cos[(2n+1)\pi x/L] \quad (\tau\ll 1). \tag{25}$$

In Fig. 2 we represent the exact SP, Eq. (23), in terms of the volatility at $x=0$ and for fixed times. The plots confirm, as hinted by Eqs. (24) and (25), that the SP decays exponentially with the volatility for both short and long times. The characteristic exponent of this decay depends on the value of θ, being larger for smaller θ, i.e., larger k [cf. Eq. (16)]. Moreover, as $\tau\gg 1$, when the volatility is small, the higher survival probability corresponds to the case when θ is smaller. This is a distinct behavior with respect to the remaining situations.

III. THE MEAN ESCAPE TIME

The survival probability $S(x,v,t)$ provides maximal information on the escape problem of the two-dimensional process $(X(t),Y(t))$. Indeed, the probability density function $f(t|x,v)$ of the escape time is related to the SP by [26]

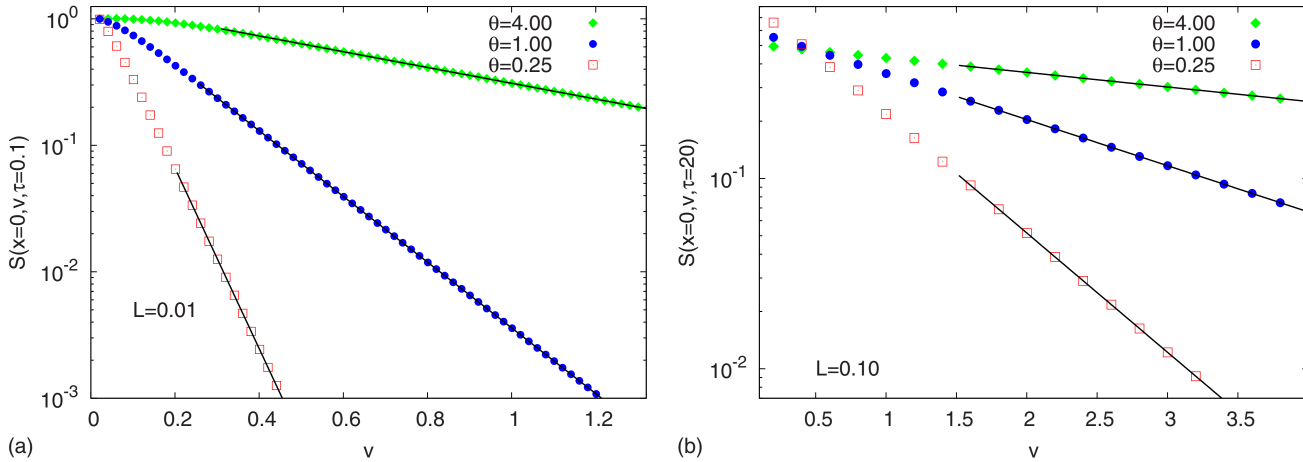

FIG. 2. (Color online) Survival probability $S(x,v,\tau)$ given by Eq. (23) at $x=0$ as a function of the volatility v. Left plot shows the case when $\tau=0.1$ ($t=2.22$ days) and $L=0.01$. The figure on the right exhibits the case when $\tau=100$ and $L=0.1$. The straight lines correspond to the exponential decay with v mentioned in the main text. Parameters of the model are $\alpha=0.045$ day^{-1}, $m=0.093$ day$^{-1/2}$, and the three different values of the parameter $\theta=(2\alpha/k^2)m^2$ provide three different values for k accordingly.

$$f(t|x,v) = -\frac{\partial S(x,v,t)}{\partial t},$$

and all moments of the escape time can be obtained through the SP. Thus, for instance, the mean escape (or exit) time is given by

$$T(x,v) = \int_0^\infty S(x,v,t)dt. \qquad (26)$$

For the Heston model we see from Eq. (23) that the two-dimensional MET, $T(x,v)$, can be written in terms of the Fourier series

$$T(x,v) = \frac{1}{\alpha}\sum_{n=0}^{\infty} T_n(v)\cos[(2n+1)\pi x/L], \qquad (27)$$

where

$$T_n(v) = \gamma_n \int_0^\infty \exp[-A_n(\tau) - B_n(\tau)v]d\tau.$$

Using Eqs. (20) and (21) and some simple manipulations, which involve the change of variable $\xi=e^{-\Delta_n\tau}$, we obtain

$$T_n(v) = \frac{\gamma_n \Delta_n^{\theta-1}}{\mu_+^\theta}\int_0^1 \frac{\xi^{-1+\mu_-\theta/\Delta_n}}{[1+(\mu_-/\mu_+)\xi]^\theta} \times\exp\left[-\mu_-\left(\frac{1-\xi}{1+(\mu_-/\mu_+)\xi}\right)v\right]d\xi. \qquad (28)$$

Figure 3 provides a three-dimensional representation of $T(x,v)$ based on the numerical computation of Eqs. (27) and (28) [28]. A noticeable aspect worth stressing is shown in Fig. 4, where two projections of the MET are depicted for either small or large volatility and also for three different values of the normal level θ. We remind the reader that we have normalized the volatility variable y with the dimensionless $v=(2\alpha/k^2)y$ [cf. Eq. (12)]. Thus when the dimensionless volatility variable v is very large, the left plot in Fig. 4 shows that the larger θ corresponds to the longer MET (in this case $v=1300$, which corresponds to $\sqrt{y}=0.2$ day$^{-1/2}$). In the opposite case of very low volatility ($v=0.001$ and $\sqrt{y}=10^{-4}$ day$^{-1/2}$) the right plot shows that this behavior is reversed, for now $\theta=1$ corresponds to a longer $T(x,v)$. This anomaly is also observed in Fig. 5 when $v<0.1$.

Having obtained the expression for $T(x,v)$ as given by Eqs. (27) and (28), let us proceed to elucidate the dependence of the MET on the volatility. This is a meaningful question from a practical point of view, for market behavior depends critically on volatility. Intuition tells us that the escape time must tend to zero as the volatility increases and a quick glance at both Eq. (28) and Fig. 3 confirms this, but what is the form of this decrease? On the other hand, the behavior of the escape time if the volatility is low is also relevant: will $T(x,v)$ grow without bound as $v\to 0$? Or will it tend to a finite, albeit maximum, value? We will next answer these questions.

Let us first obtain the behavior of the MET when $v=0$ [29]. In this case Eq. (28) reads

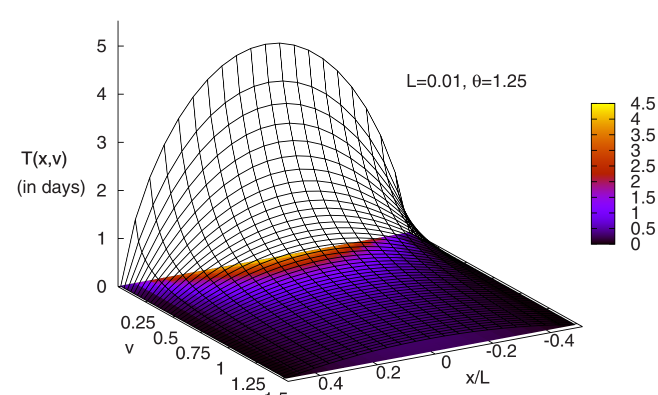

FIG. 3. (Color online) Mean escape time $T(x,v)$ given by Eq. (27) in terms of return x and volatility variable v. Parameters of the model: $\theta=1.25$, $\alpha=0.045$ day^{-1}, $m=0.093$ day$^{-1/2}$, and $k=0.0014$ day^{-1}.

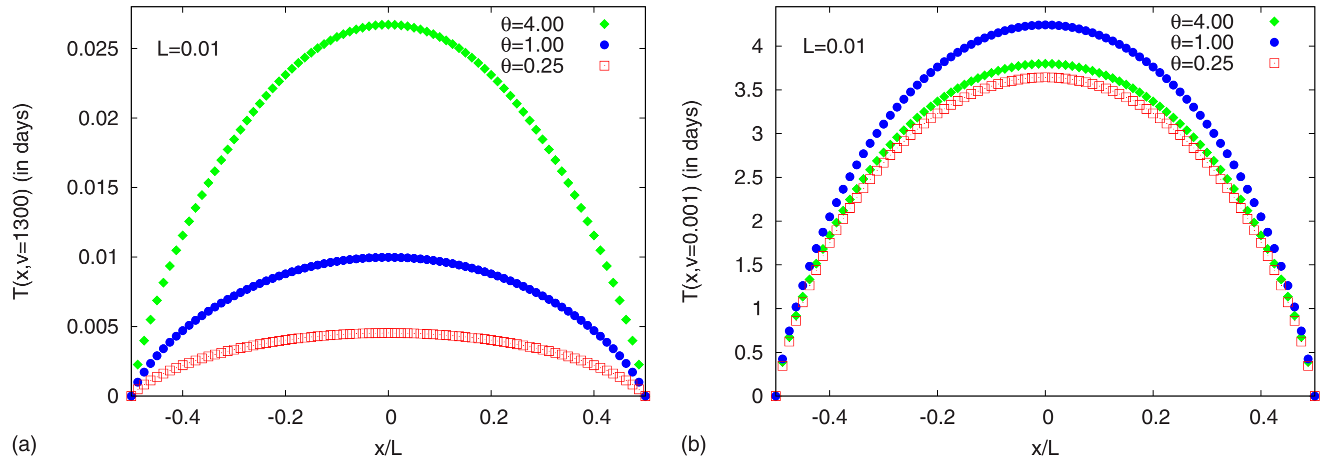

FIG. 4. (Color online) Mean escape time $T(x,v)$ given by Eq. (27) as a function of the starting return x. Left plot shows the case when v is large, showing a perfect hierarchy where larger θ means larger MET. Right plot shows how the $\theta=1$ case breaks this hierarchical order for small enough values of v. Parameters of the model are $\alpha=0.045$ day^{-1}, $m=0.093$ day$^{-1/2}$, and the three different values of the parameter $\theta=(2\alpha/k^2)m^2$ provide three different values for k accordingly.

$$T_n(0) = \frac{\gamma_n \Delta_n^{\theta-1}}{\mu_+^\theta}\int_0^1 \frac{\xi^{-1+\mu_-\theta/\Delta_n}}{[1+(\mu_-/\mu_+)\xi]^\theta}d\xi,$$

and using the integral representation of the Gauss hypergeometric function [30],

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 \xi^{b-1} \times(1-\xi)^{c-b-1}(1-\xi z)^{-a}d\xi \quad (c>b>0), \tag{29}$$

we have

$$T_n(0) = \frac{\gamma_n}{\theta\mu_-}\left(\frac{\Delta_n}{\mu_+}\right)^\theta F\left(\theta, \frac{\theta\mu_-}{\Delta_n};1+\frac{\theta\mu_-}{\Delta_n};-\frac{\mu_-}{\mu_+}\right). \tag{30}$$

We therefore see that the mean escape time $T(x,v)$ tends to a finite quantity when $v\to 0$.

Let us now turn to the case of increasing volatility. In this situation it is convenient to perform the following change of integration variable in Eq. (28):

$$z = \frac{1-\xi}{\mu_+ + \mu_-\xi},$$

then [recall that $\mu_+\mu_-=(\beta_n/2L)^2$]

$$T_n(v) = \gamma_n \int_0^{1/\mu_+} g(z)e^{-(\beta_n/2L)^2 vz}dz,$$

where

$$g(z) = (1-\mu_+ z)^{-1+\mu_-\theta/\Delta_n}(1+\mu_- z)^{-1+\theta-\mu_-\theta/\Delta_n}.$$

As $v\to\infty$ the exponential term falls off quickly, and we may safely change the upper integration limit $1/\mu_+$ by ∞. Using then Watson's lemma we write [31]

$$T_n(v) \sim \gamma_n \sum_{k=o}^{\infty} g^{(k)}(0)\frac{(2L/\beta_n)^{2k}}{v^{k+1}}. \tag{31}$$

Up to the leading order ($g(0)=1$)

$$T_n(v) \sim \gamma_n (2L/\beta_n)^2 (1/v) + O(1/v^2),$$

or [cf. Eqs. (11) and (16)]

$$T_n(v) \sim \frac{16\alpha^2 L^2}{\pi^3 k^2}\frac{(-1)^n}{(2n+1)^3}(1/v) + O(1/v^2). \tag{32}$$

Therefore,

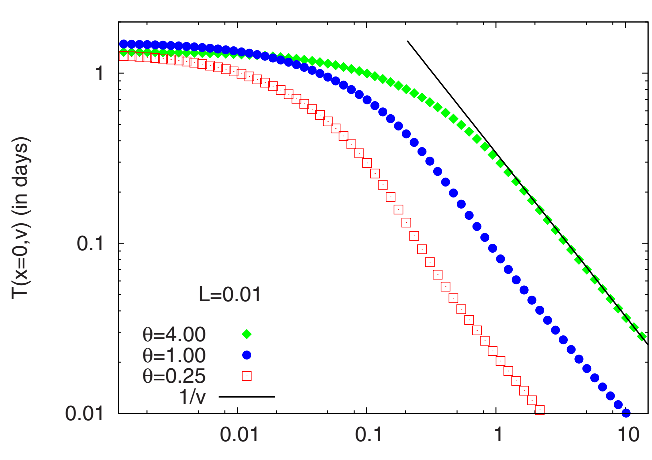

FIG. 5. (Color online) Mean escape time $T(x,v)$ given by Eq. (27) at $x=0$ as a function of the starting volatility variable v. The drawings illustrate that the MET saturates at a certain maximum value when $v=0$. On the other hand, the straight line clearly shows that the MET decays as $1/v$ with increasing volatility as we prove in the main text. Parameters of the model are the same than those of Fig. 4.

$$T(x,v) \sim \frac{16\alpha L^2}{\pi^3 k^2}(1/v)\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}\cos[(2n+1)\pi x/L] + O(1/v^2).$$

The series on the right can be summed with the result [32]

$$T(x,v) \sim \frac{2\alpha}{k^2 v}[(L/2)^2 - x^2] + O(1/v^2). \tag{33}$$

This is a remarkable result since shows that for large volatility the MET has the same form as that of the Wiener process (see Sec. IV). Moreover, the two-dimensional MET decreases linearly as $1/v$.

The behavior of $T(x,v)$ with volatility is clearly seen from the numerical evaluation of the exact MET given by Eqs. (27) and (28). Figure 5 shows, on a log-log scale, how the MET saturates to a maximum when v tends to zero, while for large volatility $T(x,v)$ is well fitted with a power law with exponent -1 which confirms the asymptotic expression (33).

IV. AVERAGING THE VOLATILITY

In real financial data the volatility is, in fact, a hidden variable which has to be measured in an indirect way [33]. It is therefore of great significance to know whether the price of an asset remains inside a given interval, regardless of its volatility. In physics the analog to this question would be knowing the survival probability for the position of a Brownian particle without worrying about its velocity [10]. Considering the entanglement between return and volatility (or position and velocity), this is certainly a difficult question and one often has to rely on approximate answers. Fortunately, the latter is not the case in the Heston model, as we shall see next.

A. The survival probability of the return

In order to obtain the SP of the return, $S(x,\tau)$, regardless the value of the volatility, we have to average the volatility away from $S(x,v,\tau)$. We will do this by assuming that, at the time we measure the return, the volatility process has reached the stationary state [34]. We therefore define $S(x,\tau)$ as the average:

$$S(x,\tau) = \int_0^{\infty} S(x,v,\tau)p_{\mathrm{st}}(v)dv, \tag{34}$$

where $p_{\mathrm{st}}(v)$ is the stationary probability density of the volatility. For the Heston model this density is the normalized solution of the Fokker-Planck equation

$$\frac{d}{dv}\left((v-\theta) + \frac{d}{dv}v\right)p_{\mathrm{st}}(v) = 0,$$

which is given by the Gamma distribution:

$$p_{\mathrm{st}}(v) = \frac{1}{\Gamma(\theta)}v^{\theta-1}e^{-v}. \tag{35}$$

Note that θ is the stationary variance of the volatility variable v. Indeed, from Eq. (35) we see at once that $\theta = \langle v^2\rangle_{\mathrm{st}} - \langle v\rangle_{\mathrm{st}}^2$. Observe also the changing shape of the stationary distribution (specially as $v\to 0$) according to whether $\theta<1$ or $\theta>1$; a fact that, as we shall see below, has consequences for the behavior of the MET.

From Eqs. (7) and (34) we get

$$S(x,\tau) = \sum_{n=0}^{\infty} S_n(\tau)\cos[(2n+1)\pi x/L], \tag{36}$$

where

$$S_n(\tau) = \int_0^{\infty} S_n(v,\tau)p_{\mathrm{st}}(v)dv,$$

which, after making use of Eqs. (17) and (35), yields

$$S_n(\tau) = \gamma_n \frac{e^{-A_n(\tau)}}{[1+B_n(\tau)]^{\theta}}. \tag{37}$$

We will write this Fourier coefficient in a more convenient form. Let us first note that by applying Eq. (20), we can write

$$e^{-A_n(\tau)} = \left(\frac{\Delta_n e^{-\mu_-\tau}}{\mu_+ + \mu_- e^{-\Delta_n\tau}}\right)^{\theta}. \tag{38}$$

On the other hand, from Eq. (21) we see that

$$1 + B_n = \frac{\mu_+(1+\mu_-) + \mu_-(1-\mu_+)e^{-\Delta_n\tau}}{\mu_+ + \mu_- e^{-\Delta_n\tau}};$$

but $1+\mu_- = \mu_+$ and $1-\mu_+ = -\mu_-$ [cf. Eq. (22)]. Hence,

$$1 + B_n(\tau) = \frac{\mu_+^2 - \mu_-^2 e^{-\Delta_n\tau}}{\mu_+ + \mu_- e^{-\Delta_n\tau}}. \tag{39}$$

Plugging Eqs. (38) and (39) into Eq. (37), we have

$$S_n(\tau) = \gamma_n\left(\frac{\Delta_n e^{-\mu_-\tau}}{\mu_+^2 - \mu_-^2 e^{-\Delta_n\tau}}\right)^{\theta}, \tag{40}$$

and therefore,

$$S(x,\tau) = \sum_{n=0}^{\infty}\gamma_n\left(\frac{\Delta_n e^{-\mu_-\tau}}{\mu_+^2 - \mu_-^2 e^{-\Delta_n\tau}}\right)^{\theta}\cos[(2n+1)\pi x/L], \tag{41}$$

which constitutes the exact expression for the SP of the return (Fig. 6).

We will now show the asymptotic time behavior of $S(x,\tau)$. We easily see from Eq. (41) that for long times, $\tau\gg 1$, the asymptotic form of the SP is

$$S(x,\tau) \simeq \sum_{n=0}^{\infty}\gamma_n\left(\frac{\Delta_n}{\mu_+^2}\right)^{\theta} e^{-\theta\mu_-\tau}\cos[(2n+1)\pi x/L] \quad (\tau\gg 1), \tag{42}$$

while for short times $\tau\ll 1$ and after taking into account [cf. Eq. (22)]

$$\mu_+^2 - \mu_-^2 e^{-\Delta_n\tau} = \Delta_n(1+\mu_-^2\tau) + O(\tau^2),$$

we get

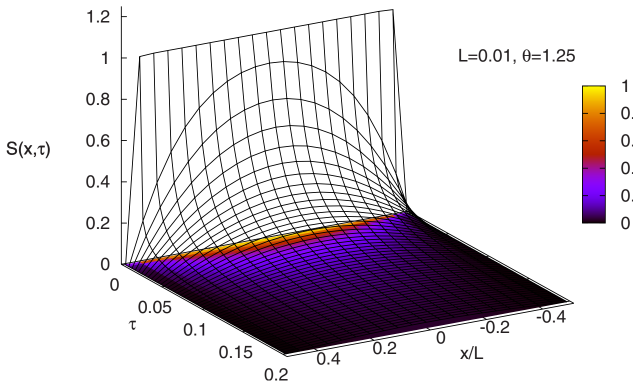

FIG. 6. (Color online) Survival probability of the return, $S(x,\tau)$, given by Eq. (41) with $L=0.01$ as a function of the return x and time $\tau=\alpha t$. Parameters of the model: $\theta=1.25$, $\alpha=0.045\ \text{day}^{-1}$, $m=0.093\ \text{day}^{-1/2}$, and $k=0.0014\ \text{day}^{-1}$.

$$S(x,\tau) \simeq \sum_{n=0}^{\infty} \gamma_n \frac{e^{-\theta\mu_-\tau}}{(1+\mu_-^2\tau)^\theta} \cos[(2n+1)\pi x/L] \quad (\tau \ll 1). \tag{43}$$

The approximate expressions for $S(x,\tau)$ given in Eqs. (42) and (43) suggest an exponential decay (essentially governed by the normal level θ) for either short and long times. This is confirmed by the numerical evaluation of the exact SP, Eq. (41), which we present in Fig. 7. We clearly see there two different exponential decays which match those shown in Eqs. (42) and (43). This exponential decay is also present for the survival probability of the hitting time as shown in Ref. [14].

B. The mean escape time of the return

In terms of the survival probability $S(x,t)$, the mean escape time is given by

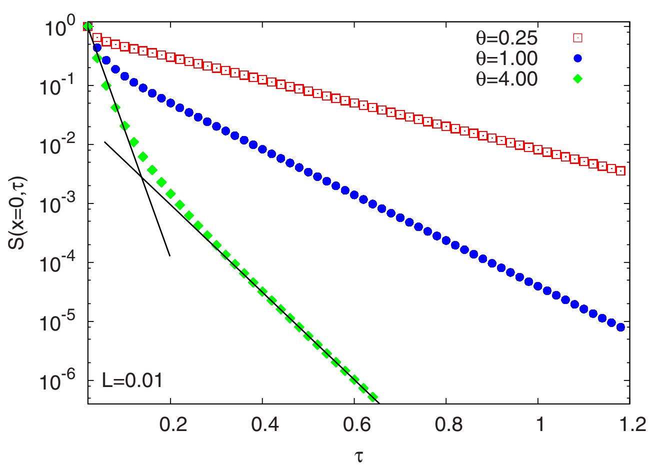

FIG. 7. (Color online) Survival probability $S(x,\tau)$ given by Eq. (41) with $L=0.01$ as a function of the time $\tau=\alpha t$ and starting from the return midpoint $x=0$. Parameters of the model are $\alpha=0.045\ \text{day}^{-1}$ and $m=0.093\ \text{day}^{-1/2}$. Notice that since $\alpha=0.045\ \text{day}^{-1}$ then $\tau=1$ corresponds to an actual time of $t\simeq 22$ days. The straight lines plotted on the lowest curve are the asymptotic approximations given by Eqs. (42) ($\tau \gg 1$) and (43) ($\tau \ll 1$).

$$T(x) = \int_0^\infty S(x,t)dt.$$

Combining this equation with Eq. (41), we see that $T(x)$ is written as a Fourier series of the form

$$T(x) = \frac{1}{\alpha}\sum_{n=0}^{\infty} T_n \cos[(2n+1)\pi x/L], \tag{44}$$

with Fourier coefficients given by

$$T_n = \gamma_n \Delta_n^\theta \int_0^\infty \left(\frac{e^{-\mu_-\tau}}{\mu_+^2 - \mu_-^2 e^{-\Delta_n \tau}} \right)^\theta d\tau. \tag{45}$$

The integral appearing in the right-hand side of this equation is evaluated by performing the change of variables $\xi=e^{-\Delta_n\tau}$. We have

$$T_n = \frac{\gamma_n \Delta_n^{\theta-1}}{\mu_+^{2\theta}} \int_0^1 \xi^{-1+\theta\mu_-/\Delta_n}[1-(\mu_-/\mu_+)^2\xi]^{-\theta} d\xi,$$

and, taking into account the integral representation of the Gauss hypergeometric function given in Eq. (29), we get

$$T_n = \frac{\gamma_n}{\theta\mu_-}\left(\frac{\Delta_n}{\mu_+^2}\right)^\theta F\left(\theta, \frac{\theta\mu_-}{\Delta_n}; 1+\frac{\theta\mu_-}{\Delta_n}; \frac{\mu_-^2}{\mu_+^2}\right).$$

Finally, the MET is given by (see Fig. 8)

$$T(x) = \frac{1}{\alpha\theta}\sum_{n=0}^{\infty} \frac{\gamma_n}{\mu_-}\left(\frac{\Delta_n}{\mu_+^2}\right)^\theta F\left(\theta, \frac{\theta\mu_-}{\Delta_n}; 1+\frac{\theta\mu_-}{\Delta_n}; \frac{\mu_-^2}{\mu_+^2}\right) \times \cos[(2n+1)\pi x/L]. \tag{46}$$

From a practical point of view, an interesting property to look at is the behavior of the MET as a function of the span L specially for short and large values of L, the latter being closely related to financial defaults or uprisings depending on the sign of x. We will thus consider the two limiting cases (a) $L\to 0$ and (b) $L\to\infty$.

(a) In the case of small span the Taylor expansion as $L\to 0$ of Eq. (46) leads to the following asymptotic expression (see Appendix B for details):

$$T(x) \sim \begin{cases} L^{\theta+1}, & \theta<1, \\ -L^2 \ln L, & \theta=1, \\ L^2, & \theta>1 \quad (L\to 0). \end{cases} \tag{47}$$

We see that in this case the behavior of the MET is governed by the (dimensionless) normal level θ which coincides with the stationary variance of the volatility variable v. Let us recall that a similar situation arises for the stationary distribution since, as seen in Eq. (35), $p_{st}(v)$ behaves in a different way according to whether the normal level is greater or lower than 1. Note that [cf. Eq. (16)] $\theta<1$ implies $m^2 < k^2/\alpha$, that is, volatility fluctuations—represented by the vol of vol k—are wilder than the tendency toward the normal level given by αm^2. On the other hand, when this tendency is greater than the volatility fluctuations (i.e., $\theta>1$) the MET grows quadratically with L independent of the normal level m but with slope depending on the vol of vol through the

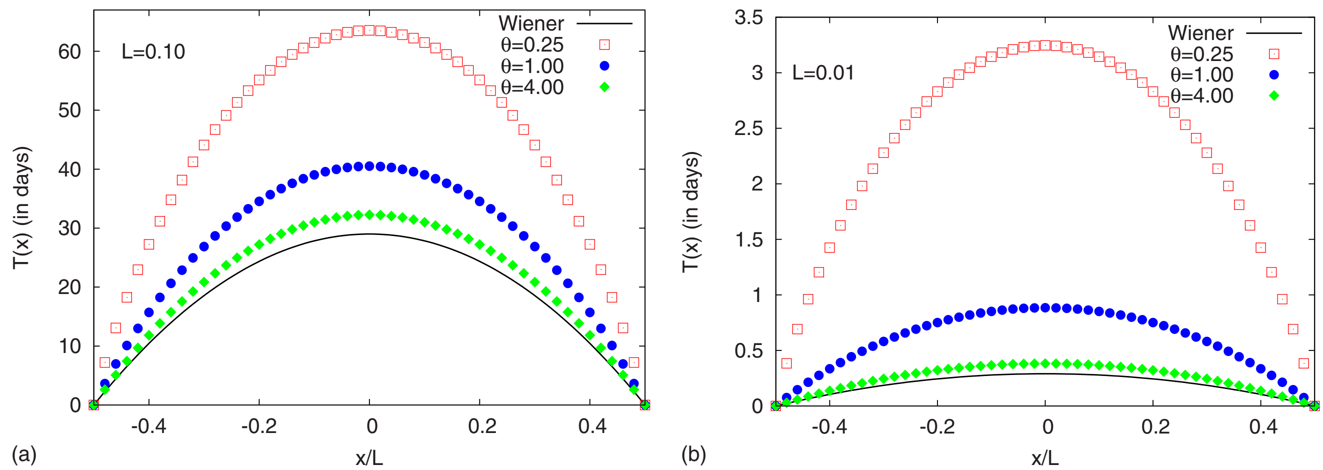

FIG. 8. (Color online) Mean escape time $T(x)$ based on the exact expression (46) as a function of the starting return x. The left figure shows the case when $L=0.1$ while the right plot shows the case when $L=0.01$. In both cases, the larger the θ the lower the MET. We also draw the MET corresponding to the Wiener process (note that the latter is always shorter than Heston's MET). Parameters of the model are the same as those of Fig. 4.

combination k^2/α [cf. Eqs. (B9) and (B10) of Appendix B]. All of this is exemplified in Figs. 9 and 10 where we plot, based on the exact expression (46), the MET as a function of the span L.

(b) Let us now look at the behavior of the MET with increasing span. Unfortunately, this case is more difficult to deal with since L appears in the Fourier series solution basically through the combination $(2n+1)/L$ and any effect due to $L\to\infty$ is neutralized by increasing values of n which, in turn, are needed to sum the Fourier series. The case (a) above turns out to be workable because the limits $L\to 0$ and $n\to\infty$ are compatible.

On the other hand, the numerical evaluation of the exact MET (46) shown in Figs. 9 and 10 clearly indicate that the MET grows quadratically with the span regardless of the value of the normal level m:

$$T(x) \sim L^2 \quad (L\to\infty). \tag{48}$$

We recall that we have already encountered this behavior at the end of Sec. III when analyzing the two-dimensional MET, $T(x,v)$, for large volatility [cf. Eq. (33)]. In Appendix C we justify this quadratic growth by means of a heuristic argument.

We also note that now, contrary to the case of small span, the slope is independent of k^2/α and all the cases which have the same m merge into a single curve (see Figs. 9 and 10).

C. The Wiener process

For many years the most ubiquitous market model has been the geometric Brownian motion which was proposed by Osborne [35]. In this model the price $P(t)$ obeys the stochastic differential equation

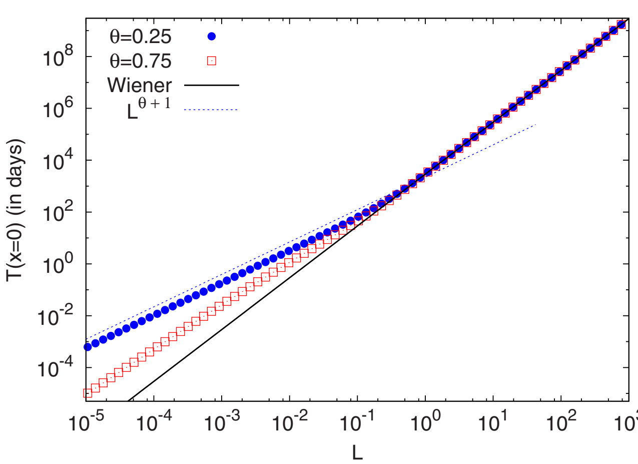

FIG. 9. (Color online) Mean escape time $T(x=0)$ given by Eq. (46) as a function of the span L when $\theta<1$. The solid lines corresponds to L^2. Parameters of the model are the same as those of Fig. 4.

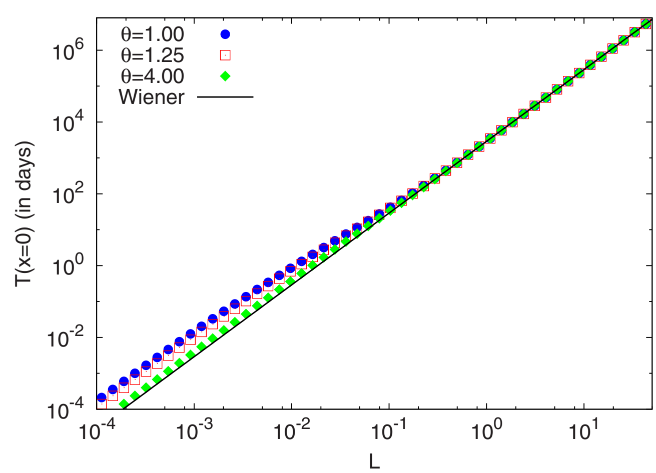

FIG. 10. (Color online) Mean escape time $T(x=0)$ given by Eq. (46) as a function of the span L when $\theta\geqslant 1$. Parameters of the model are the same as those of Fig. 4.

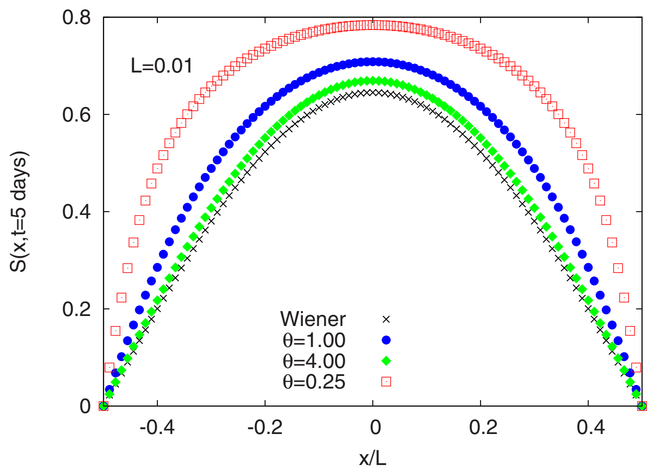

FIG. 11. (Color online) Return survival probability as a function of the scaled starting return x/L when $L=0.05$. We represent the $S(x,\tau/\alpha=5$ days) of the Heston model given by Eq. (34) for several values of θ. We also plot the SP corresponding to the Wiener process, $S_0(0,t=5$ days), given in Eq. (49). Parameters of the Heston model are $\alpha=0.045$ day^{-1} and $m=0.093$ day$^{-1/2}$. For the Wiener case we suppose that volatility is equal to the normal level $\sigma=m$.

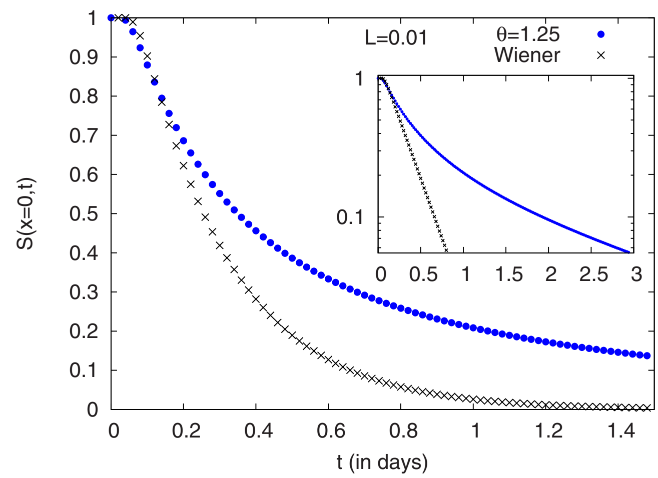

FIG. 12. (Color online) Return survival probability as a function of the scaled time τ. We represent the decay $S(0,\alpha t)$ with time for the Heston model given in Eq. (34) when $\theta=1.25$ in comparison with the Wiener model SP, $S_0(0,t)$, provided by Eq. (49). In both cases we assume that the starting return is the midpoint of the interval $L=0.01$. The inset shows same curves but on a semilogarithmic scale. Parameters of the model are the same than those of Fig. 4.

$$\frac{dP(t)}{P(t)} = \nu\, dt + \sigma\, dW(t),$$

where ν is a constant drift, σ is the volatility (a constant as well), and $W(t)$ is the Wiener process. In terms of the zero-mean return $X(t)$ defined in Eq. (1), the model reads

$$dX(t) = \sigma\, dW(t).$$

In other words, $X(t)$ is the Wiener process with variance σ^2.

In view of the widespread use of this market model among practitioners and even academicians [36], we find it convenient to compare the findings for the escape problem of the return discussed in this section with those of the Wiener process. This, in turn, may provide a test for the appropriateness of the assumption of stochastic volatility for real market models.

Let us thus suppose that the zero-mean return is described by the Wiener process and denote by $S_0(x,t)$ its survival probability inside the interval $-L/2 \leqslant X(t) \leqslant L/2$. This function obeys the equation [26]

$$\frac{\partial S_0}{\partial t} = \frac{1}{2}\sigma^2 \frac{\partial^2 S_0}{\partial x^2},$$

with initial and boundary conditions

$$S_0(x,0) = 1, \quad S_0(\pm L/2,t) = 0.$$

Proceeding as we have done before, we look for a solution to this problem in terms of a Fourier series. In this way, one easily obtains

$$S_0(x,t) = \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \exp\{-[\pi L\sigma(2n+1)]^2 t/2\} \times \cos[(2n+1)\pi x/L], \qquad (49)$$

and the MET is

$$T_0(x) = \frac{1}{\sigma^2}[(L/2)^2 - x^2]. \qquad (50)$$

In Fig. 11 we plot the $S_0(x,t)$ given by Eq. (49) in terms of the return x and for a fixed time $t=5$ days. In Fig. 12 we do the same but as a function of time and for a fixed return $x=0$. In both figures we also represent the Heston SP, $S(x,t)$, given in Eq. (41). We see that the survival probability is always higher under stochastic volatility than when the volatility is constant; although for a greater normal level θ, this difference becomes smaller.

Thus, for instance (see Fig. 12) the survival probability of the Wiener process in one day starting at $x=0$ with span $L=0.01$ is just $S_0(x=0,t=1$ day$)=0.026$; for the Heston model when $\theta=1.25$ this probability is eight times higher: $S(0,1$ day$)=0.208$. In two days the difference is even higher: $S_0(0,2$ days$)=0.0005$ versus $S(0,2$ days$)=0.095$.

This difference is also detected in the MET. Thus, in Fig. 8 we see that the Heston MET is invariably longer than that of Wiener. In other words, the Wiener process has faster escape than the Heston SV model.

Therefore, and contrary to intuition, the assumption of stochastic volatility, notwithstanding occasional bursts, seems to stabilize prices after a certain number of time steps.

D. Empirical data

Before concluding, we will consider whether the model provides realistic results in comparison with empirical data. Without the aim of being exhaustive, we have taken the Dow Jones Industrial Average (DJIA) daily index for the period between 1900 and 2004 which corresponds to 28 545 trading days. This is one of the largest daily data sets available, and we will leave for future investigations a more thorough em-

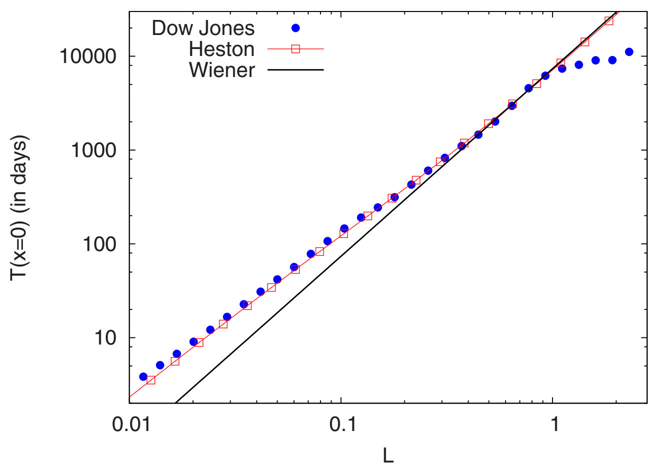

FIG. 13. (Color online) Mean escape time $T(x=0)$ as a function of the span L for the Dow-Jones Index, the Heston model (46), and the Wiener (50). Parameters of the Heston model are $m=5.8\times10^{-3}$ day$^{-1/2}$, $\alpha=0.004$ day^{-1}, $\theta=1.1$ (and $k=5.1\times10^{-4}$ day^{-1}). The Wiener case takes $\sigma=5.8\times10^{-3}$ day$^{-1/2}$.

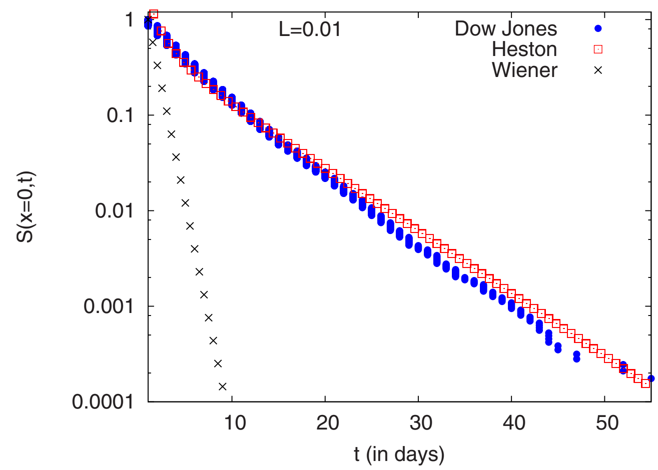

FIG. 14. (Color online) Return survival probability as a function of time for the Dow-Jones Index, the Heston model (41), and the Wiener model (49) when $L=0.012$. Parameters of the Heston model are $m=5.8\times10^{-3}$ day$^{-1/2}$, $\alpha=0.02$ day^{-1}, $\theta=1.1$ (and $k=1.2\times10^{-3}$ day^{-1}). The Wiener case takes $\sigma=5.8\times10^{-3}$ day$^{-1/2}$.

pirical study on high-frequency data and other daily data sets. The purpose of this section is simply to illustrate how feasible it is to describe real data with the theoretical Heston model.

We have first constructed the discrete version of the (daily) zero-mean return,

$$\frac{X(t+1\text{ day})-X(t)}{X(t)}=\frac{P(t+1\text{ day})-P(t)}{P(t)}-\left\langle\frac{P(t+1\text{ day})-P(t)}{P(t)}\right\rangle,$$

for later computation of the MET and the SP for $X(t)$ inside the interval $-L/2\leqslant X(t)\leqslant L/2$ and with starting return $X(0)=0$.

Figure 13 shows the MET of the DJIA index and two possible fits. The first fit corresponds to the Wiener case, Eq. (50), with $\sigma=5.8\times10^{-3}$ day$^{-1/2}$. This value is smaller than the historical standard deviation of the DJIA daily return which is equal to 7.1×10^{-3} day$^{-1/2}$. The second fit corresponds to the Heston model, Eq. (46), with parameters that keep some consistency with those provided by Refs. [20,21] for fitting the (barrier-free) probability density function of the DJIA [37]. The empirical data becomes scarce when we want to look at exit times from a span greater than 1 (which corresponds to a 100% growth rate of the index). For very small spans (corresponding to less than 1% growth rate) finite size effects emerge because of the lack of intraday data. Broadening the span domain in these two directions would require having higher-frequency data. We will leave this study for future investigations.

A similar situation is observed in the survival probability case. The Heston and Wiener SP's shown in Fig. 14 need to be renormalized at time $t=1$ day since empirical data are unable to see exit times below a one day time horizon. After this manipulation, we can observe that the Wiener SP, Eq. (49), with σ estimated from the MET plot, shows an extremely swift decay. On the other hand, the Heston model, Eq. (41), greatly improves the description of empirical data with the right exponential decay.

Finally, we mention that the results herein obtained are also qualitatively consistent with a more exhaustive view on financial data as presented in Refs. [14–17] where the hitting time (i.e., single barrier) survival probability on 1071 stocks traded at the New York Stock Exchange was studied, finding an exponential decay for large enough times in both daily [14–16] and intraday [17] data. Again, we will leave a more exhaustive test on different sources of empirical data for future research.

V. SUMMARY AND CONCLUSIONS

We have studied the escape problem of the return under the assumption of stochastic volatility given by the Heston model. The problem is fully characterized by knowing the survival probability $S(x,y,t)$ of the bidimensional process $(X(t),Y(t))$ inside the strip

$$-L/2\leqslant X(t)\leqslant L/2,\quad 0<Y(t)<\infty,$$

where $X(t)$ is the zero-mean return and $Y(t)$ is the volatility variable which, for the Heston model, is related to the volatility σ by $\sigma(t)=\sqrt{Y(t)}$. The survival probability obeys the backward Fokker-Planck equation (5) with initial and boundary conditions given in Eq. (6). We have been able to exactly solve the problem by means of the Fourier series expansion given in Eq. (23).

Once we have the solution for $S(x,y,t)$, another interesting and most useful quantity to know is the mean escape time. For the entire process given by the return and the volatility, the MET $T(x,y)$ is exactly given by a Fourier series as well [see Eqs. (27) and (28)]. We have shown that, as the volatility decreases, $T(x,y)$ tends toward a maximum, albeit finite, value. Moreover, as the volatility increases the MET decreases following the hyperbola $1/y$ which is quite remarkable because this is exactly the behavior of the MET with the volatility had the return followed the Wiener process (i.e., constant volatility) instead of the Heston model.

Real financial data consist of time series of prices and the volatility is not directly recorded and only observed in an indirect way. This hidden character makes it worth averaging out volatility from the expressions of $S(x,y,t)$ and $T(x,y)$ and thus solving the escape problem for the return alone. The assumption to be made is that the volatility has reached the stationary state; in the Heston model the latter is characterized by the Gamma distribution, Eq. (35).

Following this method, we have obtained exact expressions of $S(x,t)$, Eq. (41), and $T(x)$, Eq. (46), both in terms of Fourier series. The SP has two different exponential decays: one for long times and another, which is faster, for short times. We have been able to get analytical expressions for both decays.

We next analyzed the behavior of $T(x)$ as a function of the span L, especially for short and large values of L. The latter case is particularly significant because large values of L are associated with financial uprisings or defaults. We have shown that the behavior of the MET as $L\to 0$ depends on the normal level θ and it is given in Eq. (47). On the other hand, when $L\to\infty$ the MET grows as L^2 independently of the normal level.

Therefore, when $\theta<1$ (i.e., if volatility fluctuations are greater than the tendency toward the normal level), we have a "crossover" in the MET, from $L\to 0$ to $L\to\infty$, of the form

$$T(x)\sim L^{\theta+1}\to T(x)\sim L^2 \quad (\theta<1).$$

On the other hand, for $\theta>1$ (the tendency to relax toward the normal level is now stronger than the fluctuations of the volatility) there is no such crossover, since $T(x)\sim L^2$ for both small and large values of the span. Again, this quadratic dependence is the same as if the return had been described by the ordinary Wiener process.

We have finally compared the return SP and MET to those provided by the assumption of constant volatility. In other words, we have confronted the escape problem of the Heston model with that of the Wiener process. Our main finding is that Heston's SP is bigger and Heston's MET is longer than those corresponding to the Wiener process. This, at first sight, is contrary to intuition because a random volatility, despite occasional bursts, would seem to stabilize prices to a greater extent than a constant volatility. However, let us recall that in $S(x,t)$ and $T(x)$ the volatility has been averaged around its mean value, which is precisely the normal level θ, and, if θ is not very large, this fact may be responsible for the stabilization of the return.

A final remark. The Heston model is one among several possible candidates aimed at describing a realistic price dynamics. The question of which SV model is more appropriate as a market model is still an open question [38]. We have chosen the Heston model to carry out the present development because, as we have seen, it allows for an exact treatment. In forthcoming works we will present an approximation scheme in order to study the escape problem for a wider class of models and perform a more complete empirical analysis to better answer this question.

ACKNOWLEDGMENTS

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APPENDIX A: FUNCTIONS $A_n(\tau)$ and $B_n(\tau)$

To obtain the functions $A_n(\tau)$ and $B_n(\tau)$ we must solve the Riccati equation

$$\dot{B}_n = -B_n - B_n^2 + (\beta_n/2L)^2, \tag{A1}$$

with initial condition $B_n(0)=0$. To this end we define a new function $Z(\tau)$, related to $B_n(\tau)$ by

$$B_n(\tau) = \frac{\dot{Z}}{Z}.$$

Then $Z(\tau)$ obeys the linear equation

$$\ddot{Z} + \dot{Z} - (\beta_n/2L)^2 Z = 0,$$

whose solution reads

$$Z(\tau) = C_1 e^{\mu_- \tau} + C_2 e^{-\mu_+ \tau},$$

where C_1 and C_2 are arbitrary constants and

$$\mu_\pm = (\Delta_n \pm 1)/2, \quad \Delta_n = \sqrt{1+(\beta_n/L)^2}.$$

The expression for $B_n(\tau)$ is thus given by

$$B_n(\tau) = \frac{\mu_- e^{\mu_-\tau} - (C_2/C_1)\mu_+ e^{-\mu_+\tau}}{e^{\mu_-\tau} + (C_2/C_1)e^{-\mu_+\tau}},$$

and the initial condition $B_n(0)=0$ yields

$$C_2/C_1 = \mu_-/\mu_+.$$

Hence,

$$B_n(\tau) = \mu_- \frac{1-e^{-\Delta_n\tau}}{1+(\mu_-/\mu_+)e^{-\Delta_n\tau}}, \tag{A2}$$

which is Eq. (21).

Plugging Eq. (A2) into Eq. (18) and setting $\xi=e^{-\Delta_n s}$ as a new integration variable, we get

$$A_n(\tau) = \frac{\theta\mu_-}{\Delta_n}\int_{e^{-\Delta_n\tau}}^{1} \frac{1-\xi}{\xi[1+(\mu_-/\mu_+)\xi]} d\xi,$$

but

$$\int \frac{1-\xi}{\xi[1+(\mu_-/\mu_+)\xi]} d\xi = \ln\xi - (1+\mu_+/\mu_-)\ln[1+(\mu_-/\mu_+)\xi].$$

Hence (recall that $\mu_+ + \mu_- = \Delta_n$),

$$A_n(\tau) = \theta\left[\mu_-\tau + \ln\left(\frac{\mu_+ + \mu_- e^{-\Delta_n\tau}}{\Delta_n}\right)\right], \tag{A3}$$

which is Eq. (20).

APPENDIX B: BEHAVIOR OF THE MET FOR SMALL SPANS

Let us suppose $L \to 0$. From Eq. (22) we see that

$$\Delta_n = (\beta_n/L)[1 + (L/\beta_n)^2/2 + O(L^4)]$$

and

$$\mu_\pm = (\beta_n/2L)[1 \pm (L/\beta_n) + O(L^2)].$$

Hence,

$$\frac{\mu_-^2}{\mu_+^2} = 1 - (4L/\beta_n) + O(L^2), \tag{B1}$$

$$\frac{\mu_-}{\Delta_n} = [1 - (L/\beta_n) + O(L^2)]/2, \tag{B2}$$

and

$$\frac{1}{\mu_-}\left(\frac{\Delta_n}{\mu_+^2}\right)^\theta = 2^{2\theta+3}(L/\beta_n)^{\theta+1}[1 + (1-2\theta)(L/\beta_n) + O(L^2)]. \tag{B3}$$

(i) Suppose that $\theta < 1$. Using Eq. (B1) we write

$$F\left(\theta, \frac{\theta\mu_-}{\Delta_n}; 1 + \frac{\theta\mu_-}{\Delta_n}; \frac{\mu_-^2}{\mu_+^2}\right) = F\left(\theta, \frac{\theta\mu_-}{\Delta_n}; 1 + \frac{\theta\mu_-}{\Delta_n}; 1\right) + O(L),$$

but [30]

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\gamma(c-a)\Gamma(c-b)} \quad (c-b-a > 0). \tag{B4}$$

Note that the condition $c-b-a>0$ implies $\theta<1$. We thus find [see Eqs. (B1) and (B2)]

$$F\left(\theta, \frac{\theta\mu_-}{\Delta_n}; 1 + \frac{\theta\mu_-}{\Delta_n}; \frac{\mu_-^2}{\mu_+^2}\right) = \frac{\Gamma(1+\theta/2)\Gamma(1-\theta)}{\Gamma(1-\theta/2)}[1 + O(L)] \quad (\theta < 1). \tag{B5}$$

Plugging Eqs. (B3)–(B5) into Eq. (46) and taking into account Eqs. (11) and (16), we finally get

$$T(x) = N_1 L^{\theta+1} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^{\theta+2}} \cos[(2n+1)\pi x/L] \times [1 + O(L)] \quad (\theta < 1), \tag{B6}$$

where

$$N_1 = \left(\frac{2^{2\theta+5}}{\pi\alpha\theta}\right)\left(\frac{2\alpha}{\pi k}\right)^{\theta+1} \frac{\Gamma(1+\theta/2)\Gamma(1-\theta)}{\Gamma(1-\theta/2)}. \tag{B7}$$

For $x=0$ we have

$$T(0) = K_1 L^{\theta+1}[1 + O(L)] \quad (\theta < 1), \tag{B8}$$

where

$$K_1 = N_1 \sum_{n=0}^{\infty} (-1)^n/(2n+1)^{\theta+2},$$

and for small values of the span L the MET grows as a power law with exponent related to the normal level of the volatility $\theta < 1$.

(ii) Suppose now that $\theta > 1$. We employ the following property of the hypergeometric function [30]:

$$F(a,b;c;z) = (1-z)^{c-a-b} F(c-a, c-b; c; z),$$

and write

$$F\left(\theta, \frac{\theta\mu_-}{\Delta_n}; 1 + \frac{\theta\mu_-}{\Delta_n}; \frac{\mu_-^2}{\mu_+^2}\right) = \left(1 - \frac{\mu_-^2}{\mu_+^2}\right)^{1-\theta} F\left(1 - \theta + \frac{\theta\mu_-}{\Delta_n}, 1; \frac{\theta\mu_-}{\Delta_n}; \frac{\mu_-^2}{\mu_+^2}\right)$$

which, after use of Eq. (B1), reads

$$F\left(\theta, \frac{\theta\mu_-}{\Delta_n}; 1 + \frac{\theta\mu_-}{\Delta_n}; \frac{\mu_-^2}{\mu_+^2}\right) = \left(\frac{4L}{\beta_n}\right)^{1-\theta}\left[F\left(1 - \theta + \frac{\theta\mu_-}{\Delta_n}, 1; \frac{\theta\mu_-}{\Delta_n}; 1\right) + O(L)\right].$$

Note that we can apply Eq. (B4) since the condition $c-a-b>0$ now implies $\theta>1$. Hence [see Eq. (B2)],

$$F\left(\theta, \frac{\theta\mu_-}{\Delta_n}; 1 + \frac{\theta\mu_-}{\Delta_n}; \frac{\mu_-^2}{\mu_+^2}\right) = \left(\frac{4L}{\beta_n}\right)^{1-\theta}\left(\frac{1}{\theta-1}\Gamma(1+\theta/2) + O(L)\right).$$

Substituting this into Eq. (46) and taking into account Eq. (B3), we obtain

$$T(x) = N_2 L^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \cos[(2n+1)\pi x/L] \times [1 + O(L)] \quad (\theta > 1), \tag{B9}$$

where

$$N_2 = \left(\frac{2^7}{\pi\alpha\theta}\right)\left(\frac{2\alpha}{\pi k}\right)^2 \frac{\Gamma(1+\theta/2)}{1-\theta}. \tag{B10}$$

For $x=0$ we have

$$T(0) = K_2 L^2[1 + O(L)] \quad (\theta > 1), \tag{B11}$$

where

$$K_2 = N_2 \sum_{n=0}^{\infty} (-1)^n/(2n+1)^3.$$

Therefore, in this case the average escape time grows quadratically with the span—as if the zero-mean return had followed the simple Brownian motion—independently of the value of the normal level of volatility $\theta > 1$.

(iii) When $\theta=1$, we utilize the following series expansion of the hypergeometric function [30]:

$$F(a,b;a+b;z) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(n!)^2}[2\psi(n+1) - \psi(a+n) - \psi(b+n) - \ln|1-z|](1-z)^n,$$

which, when $z \to 1$, yields the approximation

$$F(a,b;a+b;z) = -\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\ln|1-z| + O(1).$$

Hence, as $L \to 0$,

$$F\left(1,\frac{\mu_-}{\Delta_n};1+\frac{\mu_-}{\Delta_n};\frac{\mu_-^2}{\mu_+^2}\right) = -(1/2)\ln(4L/\beta_n) + O(L\ln L),$$

whence,

$$T(x) = \frac{(4L)^2}{\alpha}\sum_{n=0}^{\infty}\frac{\gamma_n}{\beta_n^2}[-\ln(4L/\beta_n) + O(L\ln L)] \times\cos[(2n+1)\pi x/L] \quad (\theta = 1), \tag{B12}$$

and there is a logarithmic growth with the span when $\theta=1$.

The results above [see Eqs. (B6), (B9), and (B12)] are summarized in Eq. (47).

APPENDIX C: BEHAVIOR OF THE MET FOR LARGE SPANS

Let $T(x,y)$ be the MET of the joint process $(X(t),Y(t))$ out of the strip $-L/2 \leqslant X(t) \leqslant L/2$, $0 < Y(t) < \infty$. In terms of the SP $S(x,y,t)$ the MET is given by

$$T(x,y) = \int_0^{\infty} S(x,y,t)dt.$$

From Eqs. (5) and (6) we easily see that $T(x,y)$ is the solution to the boundary-value problem

$$\frac{1}{2}k^2 y\frac{\partial^2 T}{\partial y^2} - \alpha(y-m^2)\frac{\partial T}{\partial y} + \frac{1}{2}y\frac{\partial^2 T}{\partial x^2} = -1, \tag{C1}$$

$$T(\pm L/2,y,t) = 0. \tag{C2}$$

Now our heuristic argument: large values of the span L are equivalent to small values of x, but in this situation (as long as y is not too small) $\partial^2 T/\partial x^2$ is the dominant term in the left-hand side of Eq. (C1). This allows us to approximate the MET $T(x,y) \sim T_0(x,y)$, where the "outer" approximation [39] $T_0(x,y)$ is the solution to

$$\frac{1}{2}y\frac{\partial^2 T_0}{\partial x^2} = -1 \quad T(\pm L/2,y,t) = 0.$$

That is,

$$T_0(x,y) = \frac{1}{y}[(L/2)^2 - x^2],$$

and the MET grows as

$$T(x,y) \sim L^2 \quad (L \to \infty). \tag{C3}$$

[1] J.-P. Fouque, G. Papanicolau, and K. R. Sircar, *Derivatives in Financial Markets with Stochastic Volatility* (Cambdrige University Press, Cambridge, U.K., 2000).
[2] R. Cont, Quant. Finance **1**, 223 (2001).
[3] H. Risken, *The Fokker-Planck Equation* (Springer-Verlag, Berlin, 1984).
[4] S. Redner, *A Guide to First-Passage Processes* (Cambridge University Press, Cambridge, U.K., 2001).
[5] P. Wilmott, *Quantitative Finance* (John Wiley & Sons, New York, 2006).
[6] D. Lemmens, M. Wouters, J. Tempere, and S. Foulon, Phys. Rev. E **78**, 016101 (2008).
[7] J. Perelló, R. Sircar, and J. Masoliver, J. Stat. Mech.: Theory Exp. (2008) P06010.
[8] J. Masoliver and J. Perelló, Phys. Rev. E **75**, 046110 (2007).
[9] J. Masoliver and J. M. Porrà, Phys. Rev. Lett. **75**, 189 (1995).
[10] J. Masoliver and J. M. Porrà, Phys. Rev. E **53**, 2243 (1996).
[11] J. Masoliver, Phys. Rev. E **48**, 121 (1993).
[12] J. Masoliver, M. Montero, and J. Perelló, Phys. Rev. E **71**, 056130 (2005).
[13] M. Montero, J. Perelló, J. Masoliver, F. Lillo, S. Miccichè, and R. N. Mantegna, Phys. Rev. E **72**, 056101 (2005).
[14] G. Bonanno and B. Spagnolo, Fluct. Noise Lett. **5**, L325 (2005).
[15] G. Bonanno, D. Valenti, and B. Spagnolo, Eur. Phys. J. B **53**, 405 (2006).
[16] G. Bonanno, D. Valenti, and B. Spagnolo, Phys. Rev. E **75**, 016106 (2007).
[17] D. Valenti, B. Spagnolo, and G. Bonanno, Physica A **382**, 311 (2007).
[18] G. R. Jafari, M. S. Movahed, S. M. Fazeli, M. Reza Rahimi Tabar, and S. F. Masoudi, J. Stat. Mech.: Theory Exp. (2006) P06008.
[19] S. Heston, Rev. Financ. Stud. **6**, 327 (1993).
[20] A. Dragulescu and V. Yakovenko, Quant. Finance **2**, 443 (2002).
[21] A. Silva and V. Yakovenko, Physica A **324**, 303 (2003).
[22] The zero-mean return $X(t)$, which is defined through Eq. (1) and obeys the Langevin equation (2), is not to be confused with the detrended log return $R(t)=\ln[P(t)]-\mu t$, which obeys the Langevin equation $dR(t)=-(Y/2)dt+\sqrt{Y}dW_1(t)$ (in the Itô sense).
[23] We do not consider here the possible existence of cross correlation between the input noises $\xi_1(t)$ and $\xi_2(t)$. We will address in a future work the influence on the escape problem of a possible "leverage" correlation. Reference [15] has addressed this issue in a modified Heston process.
[24] W. Feller, Ann. Math. **54**, 173 (1951).
[25] J. C. Cox, J. E. Ingersoll, and S. A. Ross, Econometrica **53**, 385 (1985).
[26] C. W. Gardiner, *Handbook of Stochastic Methods* (Springer, Berlin, 1985).

[27] The convergence of the numerical computation is rather fast and we get accurate enough results by truncating the sum at $n=1000$ with an error tolerance below 0.01%. The parameters of the model chosen are quite realistic in the sense that they are close to those that capture the return probability density of Dow-Jones daily data [20]. We have taken: $\theta=1.25$, $\alpha=0.045$ day^{-1}, $m=0.093$ day$^{-1/2}$, and $k=0.0014$ day^{-1}, and we have thus considered the case with an interval $L=0.01$ and when $\tau=\alpha t=0.1$ ($t=2.22$ days).

[28] The numerical computation is in this case a bit more complicated than that of Fig. 1 since it requires the numerical evaluation of the integral (28), but in all situations we have found a rather fast convergence.

[29] Note that, if at a given instant of time $v=0$, the return is then frozen but only during one time step, since immediately afterward the random dynamics of the volatility is reassumed [see Eqs. (2) and (3) and recall that $m\neq 0$]. In other words, $v=0$ (i.e., $y=0$) is not a fixed point of Eq. (3).

[30] W. Magnus, F. Oberhettinger, and R. P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics* (Springer-Verlag, New York, 1966).

[31] A. Erdélyi, *Asymptotic Expansions* (Dover, New York, 1956).

[32] In effect, the Fourier coefficient of $(2\alpha/k^2v)[(L/2)^2-x^2]+O(1/v^2)$ is precisely that of Eq. (32).

[33] Z. Eisler, J. Perelló, and J. Masoliver, Phys. Rev. E **76**, 056105 (2007).

[34] This assumption is correct provided that the process started, for the first time, in the infinite past.

[35] M. Osborne, Oper. Res. **7**, 145 (1959).

[36] J. Y. Campbell, A. W. Lo, and A. Craig MacKinlay, *The Econometrics of Financial Markets* (Princeton University Press, Princeton, NJ, 1997).

[37] It is required to take $\theta=1.1$, which is consistent with the value provided in Ref. [21], but we need to take $\alpha=0.004$ day^{-1}, which is more than an order of magnitude smaller than the values provided in Refs. [20,21].

[38] J. Masoliver and J. Perelló, Quant. Finance **6**, 423 (2006).

[39] C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (McGraw-Hill, New York, 1978).